MASA student creative thinking quest 2017

The pages which follow show what a teacher might do whilst modelling project behaviour in the classroom and meeting the requirements of the Mathematics syllabus.

Mathematics is a set of processes for constructing, testing and manipulating representations of the quantitative or spatial characteristics of what we perceive.

Performing a mathematical task involves a not necessarily linear combination of experiencing, investigating, conjecturing, believing and proving aspects of the situation from which the task has emerged. (After Jon Roberts 1997)

A problem is a task for which there is no immediately obvious solution, method for finding the solution or way to investigate the situation presented.

In teaching and learning, a project involves combining Mathematics with one or more particular applications to bring together an abstraction and a situation from which it arises or to show how a particular mathematical model arises from several contexts, for example, projectile motion and the inverse square law.

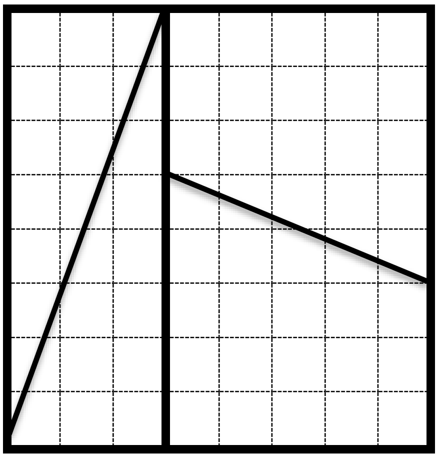
In assessment, a project requires the student to apply, discover or rediscover some Mathematics in order to use it in investigating or solving a problem.

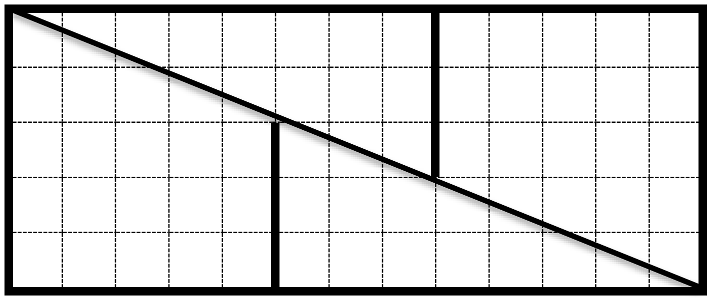
In a competition, a project requires the student, alone or as a member of a group of specified size, to undertake an investigation, possibly using concrete materials, to formulate a problem in mathematical terms, solve it and relate the solution to the original context.

Projects often involve the student work away from the teacher's presence. Having the student keep a journal, answer questions about the task and methods used and observing the development of the project can assist the teacher with having confidence that the product presented is the student's own.

Is 64 = 65?

An old puzzle involves dissecting an 8 x 8 square as in the diagram below and rearranging the pieces to form a 13 x 5 rectangle appearing to give the paradoxical result that 64 = 65.





This result prompts questions such as:

- How does this come about and are there other squares with integer length dissected into pieces with integer length edges perpendicular which give a similar result?
- How can you dissect a square into four pieces as in the diagram, but with no edges required to be of integer length, and form a rectangle with the same area?
- Are there other ways of dissecting a square into four, or more, pieces which can be arranged to preserve area or to give the same paradoxical result?

It is possible to investigate the first question using trial and error with squares of different integer edge lengths.

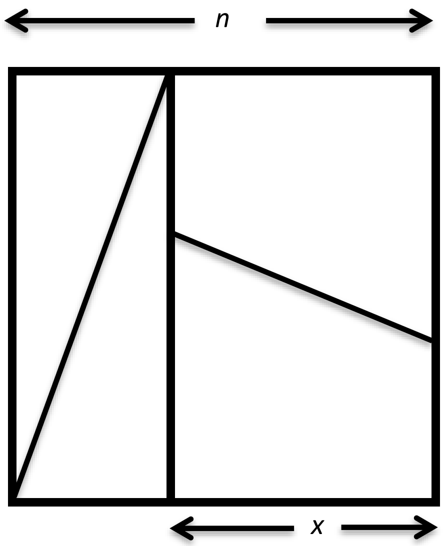
Calculators and spreadsheets can speed-up progress.

It is worthwhile providing the students with two copies of the 8 x 8 square and having them cut out both and glueing the original and rearranged dissection in their books.

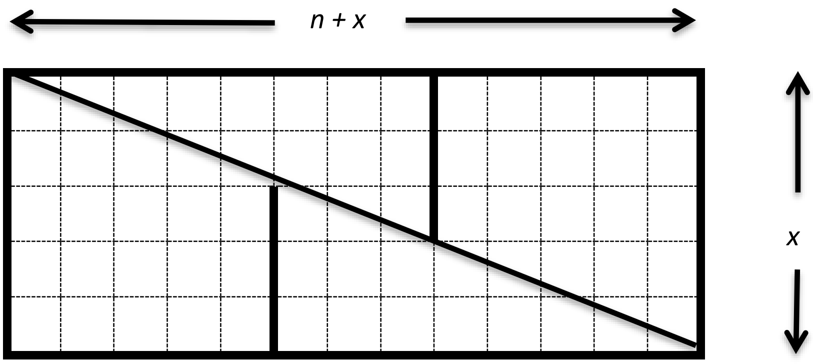
When others are found, the students can draw their own to cut out and rearrange.

Having their own diagrams and cut-outs in their books helps with explaining the apparent paradox.

It can also be approached more analytically where the square has length *n* units and the vertical cut is made *x* units from the right hand edge.



When the pieces are rearranged the rectangle has length *n + x* units and height *x* units.



The area of the rectangle being one unit more than that of the square leads to:

$x(n+x) = n^2 + 1$ where *n* is given and we wish to find *x*.

The resulting quadratic equation in *x* is:

$$x^2 + nx - (n^2 + 1) = 0$$

Using the formula for the zeros of a quadratic gives:

$$x = \frac{-n \pm \sqrt{n^2 - (-4)(n^2+1)}}{2} = \frac{-n \pm \sqrt{5n^2+4}}{2}$$

Since both *n* and *x* are integers, $5n^2 + 4$ must be a perfect square.

From the introductory problem, n = 8 should be one solution and substituting n = 8 yields:

$$x = \frac{-8 \pm \sqrt{5 \times 8^2 + 4}}{2} = \frac{-8 \pm \sqrt{324}}{2} = \frac{-8 \pm 18}{2} = 5 \; or -13$$

x = 5 is the expected answer. x = -13 might be of interest later.

Although not proof in the mathematical sense it shows that this might be a productive line to follow.

Some students might have realised that n = 1 gives a perfect square but the result is trivial.

A little leading will show that n = 3 gives a perfect square.

The 3 x 3 square can be dissected and rearranged to give a 5 x 2 rectangle.

The next largest square to behave in this paradoxical manner has edge length 21 and the resulting value of x is:

$$\frac{-21 \pm \sqrt{5 \times 21^2 + 4}}{2} = \frac{-21 \pm \sqrt{2209}}{2} = \frac{-21 \pm 47}{2} = 13 \; or -34$$

The square has area $21^2 = 441 \; units$ whilst the rectangle has area $(21 + 13) \times 13 = 442 \; units$

The values of *n* and *x* which have been used and generated as edge lengths of the squares and rectangles so far are: 1, 2, 3, 5, 8, 13, 21, 34

For some there will now be an ***aha moment*** with the realization that each term from the third is the sum of the previous two: the Fibonacci sequence!

To be consistent with the conventional presentation of the Fibonacci sequence, another 1 will be shown on the left, giving: 1, 1, 2, 3, 5, 8, 13, 21, 34

The squares for which this dissection yields a rectangle with area one unit more than the square are those with length an even numbered term in the sequence.

Is this really correct?

Consider five consecutive terms of the Fibonacci sequence, represented as follows.

x, n, n + x, 2n + x, 3n + 2x where n is an odd numbered term.

The square with edge length *n* is rearranged into a rectangle of area x(n + x)

This gives the equation: $x(n + x) - n^2 = 1$

which can be rearranged to give: $x^2 + xn - n^2 = 1$

The next odd numbered term after *n* is *2n + x*

and hence the difference between the areas of the rectangle and square is:

$$(n + x)(3n + 2x) - (2n + x)^2$$

Simplifying this expression gives: $3n^2 + 5\,nx + 2x^2 - 4n^2 - 4nx - \ x^2$

which simplifies to: $x^2 + xn - \ n^2$

In the initial example this expression had a value of 1 as is the case when *n* = 8, *x* = 5.

Neither *x* nor *n* have changed their values so the area of the rectangle must still be one more than that of the square.

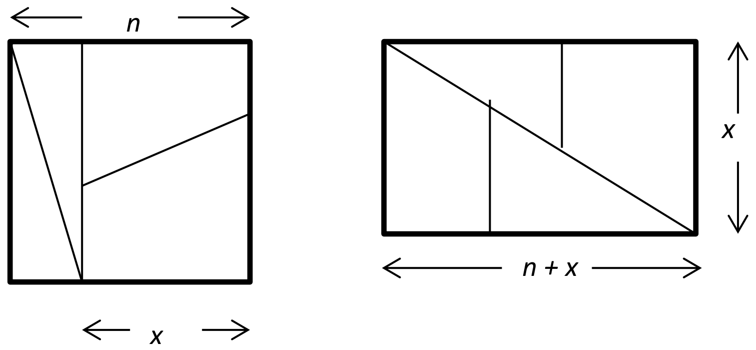
Note that these findings can be established more formally in terms of the method of mathematical induction.

Examples using squares of edge length 21, 55, 144 can be shown to follow the pattern.

What about squares of edge length 5, 13, 34, 89?

Go and try them and note what you discover.

Is there a dissection like these in which the area is preserved?



If the square and rectangle have the same areas then

$$n^2 = x(n + x)$$
$$\therefore\ x^2 + nx - n^2 = 0$$
$$\therefore x = \frac{-n \pm \sqrt{n^2 + 4n^2}}{2}$$
$$\therefore x = \frac{n(\sqrt{5} - 1)}{2}$$

x is smaller than *n* and so it is possible to dissect a square into two pairs of pieces which can be rearranged this way to form a rectangle with the same area

The number $\frac{\sqrt{5}-1}{2}$ is the golden ratio!

The area of the rectangle is $\frac{n(\sqrt{5}-1)}{2} \times \left(n + \frac{n(\sqrt{5}-1)}{2}\right) = n^2\left(\frac{(\sqrt{5}-1)}{2}\right)\left(1 + \frac{(\sqrt{5}-1)}{2}\right)$

$$= n^2\left(\frac{\sqrt{5}-1}{2}\right)\left(\frac{\sqrt{5}+1}{2}\right) = n^2$$

This is the area of the square.

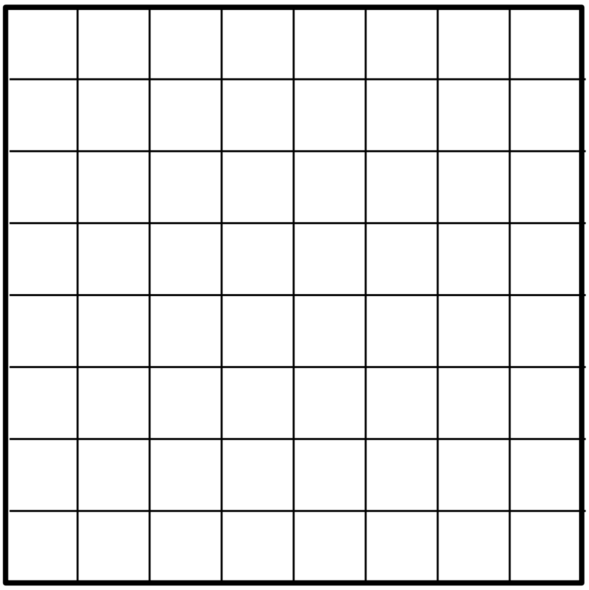
Binet's formula is a direct way for calculating the n^{th} Fibonacci number:

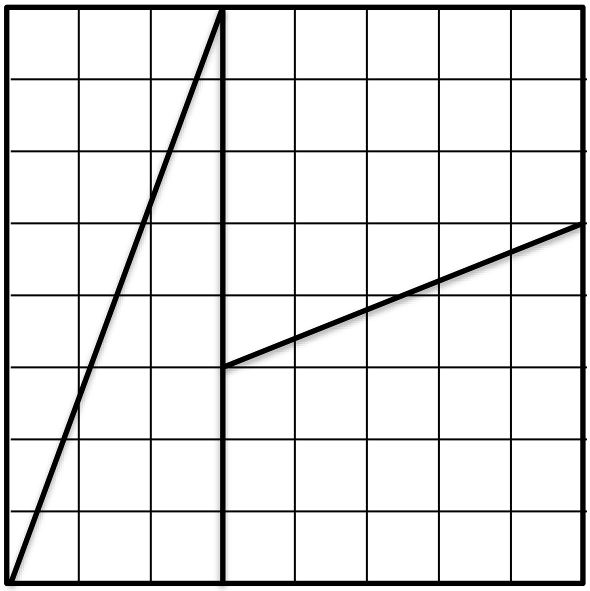
$$F_n = \frac{1}{\sqrt{5}}\left(\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n\right)$$

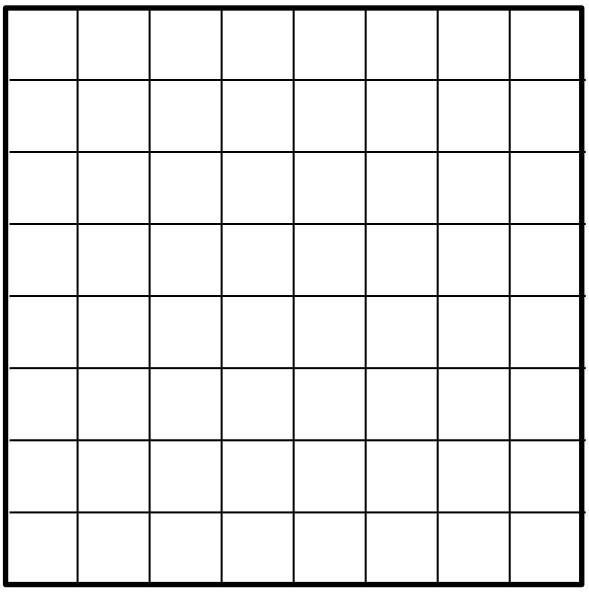
This can be used to verify the conjectures made about Fibonacci numbers, for example:

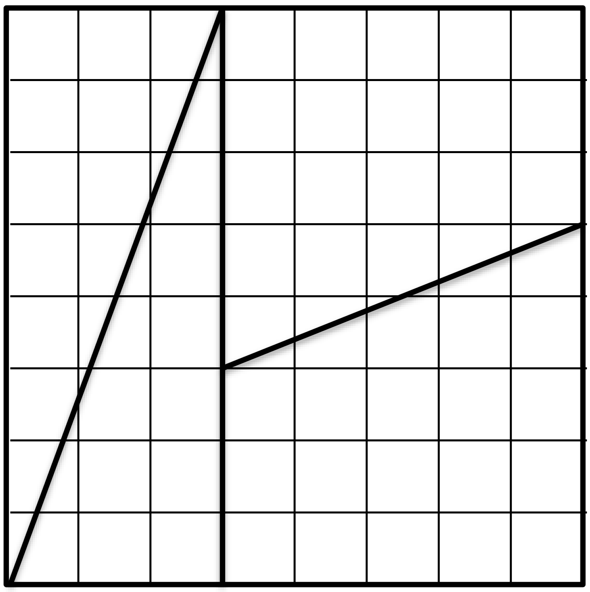
$$F_{n+1} = F_n + F_{n-1} \text{ and } F_n^2 = F_{n-1} \times F_{n+1} \pm 1$$

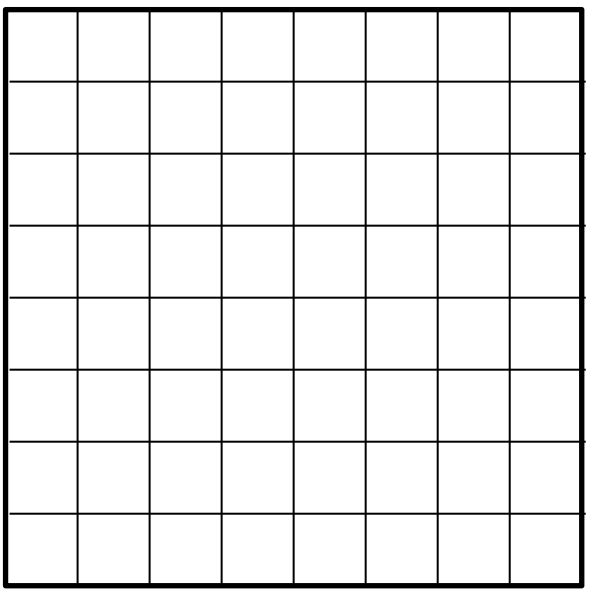
Cut outs for paradox

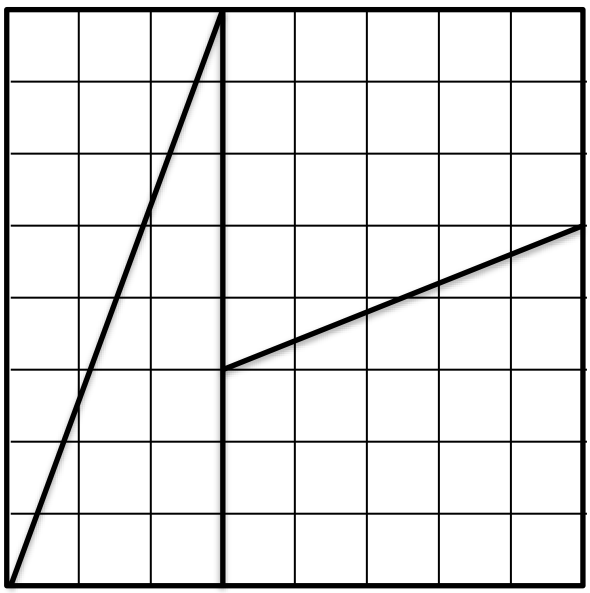












Some project ideas

Students' opinions by year level

Edge squares and other tile designs

Geometrical representations of powers of integers

Lake Eyre Canal

Insurance models

Spanning trees

Best bicycle gear

Birthday problems

Distribution of the numbers of sultanas in a batch of biscuits given the average

Collecting a full set

Reflection and refraction

Harvesting a plantation

The two second safety rule in driving